Classification and versal deformation of generalized flags

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Abstract

A natural equivalence relation can be considered in the generalized flag manifold. First we give a complete set of invariants of it as well as a canonical matrix description of the classes. Next we consider parametric flags. We give a miniversal deformation for the above canonical form and we use it to characterize the stable flags.

Keywords: Versal deformation, orbit space, Grassmann manifold, generalized flag manifold.

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Introduction

The generalized flag manifold is a generalization of the Grassmann manifold. Namely, we fix a reference flag (a chain of subspaces $F_1 \subset \cdots \subset F_k$ of a $\mathbb{F}$-vector space F) and we consider the set of flags of fixed dimension whose elements are contained in the elements of the reference flag. One can prove that this set is a smooth manifold with a similar structure to that of Grassmann manifold (see for example [1]).

The generalized flag manifold arises in a natural way while studding the topology of the set of A-invariant and (C, A)-invariant subspaces. More precisely in [1] and [2] it is proved the existence of deformation retracts of the set of A-invariant subspaces and (C, A)-invariant subspaces, respectively, on a generalized flag manifold (when the discrete invariants of the restrictions are fixed). In fact one can prove that the sets of A and (C, A) invariant subspaces are the total spaces of vector bundles on a generalized flag manifold (see [4]). Therefore, an equivalence relation defined in the set of A or (C, A)-invariant subspaces induce an equivalence relation in the generalized flag manifold. In most cases this equivalence relation is the natural one. Namely, two flags are equivalent if there exists an automorphism rending fixed the reference flag such that sends the elements of the first flag to the elements of the second one.

Our first goal is to give a geometric construction of a canonical basis for a generalized flag. From it it can derived a complete set of invariants of the corresponding class as well as a canonical form of its matrix representation (section 2). Moreover if those invariants are constant for a family of flags, one can obtain a family of basis reducing (locally) the matrix representation of the flags to its canonical form theorem 2.7.

Second we consider general perturbations of a flag and we compute a miniversal deformation for the above canonical form. In order to do that we apply the main results of [3], where it is considered a general orbit space M/Γ and an equivalence relation defined on it by means of the action of a group G. In [3] it is related a versal deformation of an element of $x \in M$ with regard a suitable group action of $G \times \Gamma$ with a versal deformation of its orbit $x\Gamma$ with regard the action of G. Then, applying Arnold's techniques the authors of [3] prove theorem 3.2 which we shall use in theorem 3.4.

In this paper we make use of the following notation. $\mathbb{F}$ is the field of either the complex or the real numbers. $M_{p,q}$ denotes the set of $p \times q$ matrices with entries in $\mathbb{F}$ and $M^*_{p,q}$ the set of the full rank ones. $M^*_{p,p}$ is the linear group $\mathrm{Gl}(p)$. If E is a vector space, $\mathrm{Gr}_d(E)$ denotes the Grassmann manifold of d-dimensional subspaces of E. Throughout the paper, we will denote by I the identity element of a group. If M is a manifold and $x \in M$, (M, x) denotes an open neighbourhood of x. We say that a basis is *adapted* to a set of subspaces if one can obtain bases of those subspaces taking subsets of this basis. In particular, a basis adapted to a chain of subspaces $V_1 \subset \cdots \subset V_k$ is a basis of V_k obtained by extending successively bases of $V_1, V_2, \ldots$. We say that an n-tuple of integers $(k_1, \ldots, k_n)$ is a *partition* if $0 \leq k_i \leq k_{i+1}$ for all $i = 1..n-1$. If $k = (k_1, \ldots, k_n)$, $h = (h_1, \ldots, h_n)$ are two partitions, if $h_i \leq k_i$ for $i = 1..n$ we put $h \prec k$. I_a means the $a \times a$-identity matrix and $0_{a,b}$ the $a \times b$-zero matrix. The non-specified entries of a displayed matrix are 0.

1 The generalized flag manifold

Let F be a fixed $\mathbb{F}$-vector space of dimension n. Let $s = (s_1, \ldots, s_k)$ be a partition with $s_k \leq n$. We call an *s-flag* of F to a chain of subspaces $V_1 \subset \cdots \subset V_k$ of F where $\dim V_i = s_i$, $i = 1, \ldots, k$. Let us denote by $\mathrm{Flag}(s)$ the set of all s-flags of F. It is clear that when $k = 1$, $\mathrm{Flag}(s)$ is, simply, the Grassmann manifold of subspaces of F of dimension s_1. One can easily check that

$$\mathrm{Flag}(s) = M^*_{n,s_k}/\Gamma$$

Where Γ is the set of full rank matrices of the form

$$P = \begin{bmatrix} P_{1,1} & P_{1,2} & \dots\dots & P_{1,k} \\ 0 & P_{2,2} & \dots\dots & P_{2,k} \\ 0 & 0 & \dots\dots & \dots \\ \dots & \dots & \dots\dots & \dots \\ 0 & 0 & \dots\ 0 & P_{k,k} \end{bmatrix}$$

with $P_{i,j}$ an $(s_i - s_{i-1}) \times (s_j - s_{j-1})$-matrix ($s_0 := 0$), and defining the action of Γ on M^*_{n,s_k} by $(X, P) \mapsto XP$. From a geometric point of view, given an s-flag $(V_1, \dots, V_k)$, V_i is spanned by the s_i-first columns of a matrix $X \in M^*_{n,s_k}$ and the elements of Γ represent changes of bases of V_{s_k} rending invariant V_{s_i}, $i = 1, \dots, k$.

In many context appears the concept of *generalized flag* (see [1]). Let $r = (r_1, \dots, r_k)$ be a partition and let $F_1 \subset \dots \subset F_k = F$ be a reference flag with $\dim F_i = r_i$. We can take a basis $\{e_1, \dots, e_{r_k}\}$ of F such that $F_i = \operatorname{span}\{e_1, \dots, e_{r_i}\}$. Let $s = (s_1, \dots, s_k)$ be a partition such that $s_i \le r_i$, $i = 1, \dots, k$. We call a *generalized s-flag* of $(F_1, \dots, F_k)$ a chain of subspaces $V_1 \subset \dots \subset V_k$, $V_i \subset F_i$, such that $\dim V_i = s_i$, $i = 1, \dots, k$.

$$\begin{array}{rccccccccc} \text{reference flag} & F_1 & \subset & F_2 & \subset & \cdots & \subset & F_k & \subset & \mathbb{F}^n \\ & \cup & & \cup & & & & \cup & & \\ \text{generalized flag} & V_1 & \subset & V_2 & \subset & \cdots & \subset & V_k & \subset & \mathbb{F}^n \end{array}$$

Let us denote by $\operatorname{Flag}(r, s)$ the set of all generalized s-flags of $(F_1, \dots, F_k)$. It is clear that if $r = (n, \dots, n)$, $\operatorname{Flag}(r, s) = \operatorname{Flag}(s)$. The geometry of $\operatorname{Flag}(r, s)$ is studied in [1]. In particular the authors of [1] prove that $\operatorname{Flag}(r, s)$ has a smooth manifold structure diffeomorphic to the orbit space

$$\mathcal{M}/\Gamma$$

where $\mathcal{M}$ is the set of full rank matrices of the form

$$X = \begin{bmatrix} X_{1,1} & X_{1,2} & \dots\dots & X_{1,k} \\ 0 & X_{2,2} & \dots\dots & X_{2,k} \\ 0 & 0 & \dots\dots & \dots \\ \dots & \dots & \dots\dots & \dots \\ 0 & 0 & \dots\ 0 & X_{k,k} \end{bmatrix}$$

with $X_{i,j}$ an $(r_i - r_{i-1}) \times (s_j - s_{j-1})$-matrix ($r_0 := 0$), and defining the action of Γ on $\mathcal{M}$, as above, by $(X, P) \mapsto XP$. The geometric interpretation of the columns of

$X \in \mathcal{M}$ is identic to the non generalized case and the 0's of those matrices are due to the condition $V_i \subset F_i$.

Remark 1.1 let be $s \prec r' \prec r$ Then the inclusion $\mathrm{Flag}(r', s) \subset \mathrm{Flag}(r, s)$ is an embedding. In particular $\mathrm{Flag}(r, s)$ is a submanifod of $\mathrm{Flag}(s)$

2 Classification of generalized flags

We define in $\mathrm{Flag}(r, s)$ the following equivalence relation.

Definition 2.1 $(V_1, \ldots, V_k) \sim (V_1', \ldots, V_k')$ *if there exist* $P \in \mathrm{Gl}(n)$ *such that*

$$V_i' = P(V_i) \text{ and } P(F_i) = F_i \text{ for all } i = 1, \ldots, k.$$

Remark 2.2 In $\mathrm{Flag}(s)$ the above equivalence relation is trivial (there is a single class). More generally, let be $s \prec r' \prec r$ Then, $(V_1, \ldots, V_k) \sim (V_1', \ldots, V_k')$ in $\mathrm{Flag}(r', s)$ implies that $(V_1, \ldots, V_k) \sim (V_1', \ldots, V_k')$ in $\mathrm{Flag}(r, s)$.

Let $\mathcal{G} = \{P \in \mathrm{Gl}(n) | \, P(F_i) = F_i\}$. It is clear that $\mathcal{G}$ is a subgroup of $\mathrm{Gl}(n)$ and that the classes of the above equivalence relation are the orbits of the generalized flags with regard to the action of $\mathcal{G}$ on $\mathrm{Flag}(r, s)$ defined by

$$(P, (V_1, \ldots, V_k)) \mapsto (P(V_1), \ldots, P(V_k)).$$

One can easily check that $\mathcal{G}$ is the set of full rank matrices of the form

$$P = \begin{bmatrix} P_{1,1} & P_{1,2} & \ldots\ldots & P_{1,k} \\ 0 & P_{2,2} & \ldots\ldots & P_{2,k} \\ 0 & 0 & \ldots\ldots & \ldots \\ \ldots & \ldots & \ldots\ldots & \ldots \\ 0 & 0 & \ldots \; 0 & P_{k,k} \end{bmatrix}$$

with $P_{i,j}$ an $(r_i - r_{i-1}) \times (r_j - r_{j-1})$-matrix

It is clear that there exists a bijection between the orbits $\mathcal{G}(X\Gamma)$, $X\Gamma \in \mathcal{M}/\Gamma$, and the orbits $\mathcal{G}X\Gamma$ corresponding to the action on $\mathcal{M}$ defined by

$$((P, Q), X) \mapsto PXQ$$

where in $\mathcal{G} \times \Gamma$ we define the group product $(P, Q)(P', Q') := (P'P, QQ')$.

Now we will find a canonical element of the orbit $\mathcal{G}X\Gamma$. According to the geometric interpretation of the elements of $\mathcal{M}, \mathcal{G}$ and Γ, we will find a basis of F adapted to the flag $(F_1, \dots, F_k)$ by extending a basis of V_k adapted to the flag $(V_1, \dots, V_k)$. In order to do that, let us consider the diagram

$$
\begin{array}{ccccccccc}
V_1 = V_1 \cap F_1 & \subset & V_2 \cap F_1 & \subset & \cdots & \subset & V_k \cap F_1 & \subset & F_1 \\
\cap & & \cap & & & & \cap & & \cap \\
V_2 & = & V_2 \cap F_2 & \subset & \cdots & \subset & V_k \cap F_2 & \subset & F_2 \\
& & \cap & & & & \cap & & \cap \\
& & V_3 & = & \cdots & \subset & V_k \cap F_3 & \subset & F_3 \\
& & & & \ddots & & \vdots & & \vdots \\
& & & & & & V_k & \subset & F_k
\end{array} \tag{1}
$$

Defining $F_{k+1} = V_{k+1} = F$ and $F_0 = V_0 = \{0\}$, we can write for $k \geq i \geq j \geq 0$

$$
\begin{array}{ccc}
V_i \cap F_j & \subset & V_{i+1} \cap F_j \\
\cap & & \cap \\
V_i \cap F_{j+1} & \subset & V_{i+1} \cap F_{j+1}
\end{array}
$$

where, obviously, $V_i \cap F_j = (V_{i+1} \cap F_j) \cap (V_i \cap F_{j+1})$. Let $E_{i,j}$ be a subspace such that

$$V_i \cap F_j = E_{i,j} \oplus (V_{i-1} \cap F_j + V_i \cap F_{j-1})$$

and let $e_{i,j}$ be a basis of $E_{i,j}$. We have that, for $0 < i \leq k$,

$$
\begin{aligned}
V_i &= \operatorname{span}(e_{1,1} \cup e_{21} \cup e_{2,2} \cup \cdots \cup e_{i,1} \cup \cdots \cup e_{i,i}) \\
F_i &= \operatorname{span}(e_{1,1} \cup \cdots \cup e_{k+1,1} \cup \cdots \cup e_{i,i} \cup \cdots \cup e_{k+1,i})
\end{aligned}
$$

Now, we arrange the vectors of $\bigcup_{i,j} e_{i,j}$ in order to obtain bases of V_k and F_k in the following way

$$
\begin{array}{ll}
\{e_{1,1};\ e_{2,1}, e_{2,2}; \dots; e_{k,1}, \dots, e_{k,k}\} & \text{basis of } V_k \text{ adapted to } (V_1, \dots, V_k) \\
\{e_{1,1}, \dots, e_{k,1};\ e_{2,2}, \dots, e_{k,2}; \dots; e_{k+1,k+1}\} & \text{basis of } F_k \text{ adapted to } (F_1, \dots, F_k)
\end{array}
$$

Arranging in columns the coefficient of the elements of the first basis with regard the second one can check that we obtain the following matrix

(2) $X = [X_{j,i}]$, $0 < j \leq i \leq k+1$, with $X_{j,i} = 0$ if $i < j$ and otherwise,

$$X_{j,i} = \begin{bmatrix} 0_{\beta_{i,j},\alpha_{i,j}} & 0 & 0 \\ 0 & I_{s_{i,j}} & 0 \\ 0 & 0 & 0_{\overline{\beta}_{i,j},\overline{\alpha}_{i,j}} \end{bmatrix}$$

where

$$s_{i,j} := \dim E_{i,j} = \dim V_i \cap F_j - \dim V_{i-1} \cap F_j - \dim V_i \cap F_{j-1} + \dim V_{i-1} \cap F_{j-1}$$

$$\alpha_{i,j} := s_{i-1,j} + s_{i-2,j} + \cdots s_{j,j} \text{ or } 0 \text{ if } i = j$$
$$\overline{\alpha}_{i,j} := s_{k+1,j} + \cdots + s_{i+1,j}$$

$$\beta_{i,j} := s_{i,j-1} + \cdots + s_{i,1} \text{ or } 0 \text{ if } j = 0$$
$$\overline{\beta}_{i,j} := s_{i,i} + \cdots + s_{i,j+1}$$

Remark 2.3 If $X \in \mathcal{M}$ represents the flag $(V_1, \ldots, V_k)$, the sizes of its blocks, $X_{i,j}$, are

$$s_i - s_{i-1} = s_{i,j} + \beta_{i,j} + \overline{\beta}_{i,j} = s_{i,1} + \cdots + s_{i,i}$$
$$r_j - r_{j-1} = s_{i,j} + \alpha_{i,j} + \overline{\alpha}_{i,j} = s_{j,j} + \cdots + s_{k,j}$$

Remark 2.4 Since $\dim V_i \cap F_j$ for $0 < j \leq i \leq k$ is clearly a set of invariants of the flag $(V_1, \ldots, V_k)$, the set of integers $s_{i,j}$ is also a set of invariants of the above flag. Moreover, given a set of integers $s_{i,j}$, $0 < j \leq i \leq k$, satisfying the conditions of remark 2.3, there exists a flag $(V_1, \ldots, V_k)$ having those invariants.

We have proved the following theorem

Theorem 2.5 *Let $(V_1, \ldots, V_k)$ be a generalized flag of $(F_1, \ldots, F_k)$ represented by $X \in \mathcal{M}$. With the above notations we have that*

(1) The matrix defined in (2) is a canonical form of the orbit $\mathcal{G}X\Gamma$.
(2) $s_{i,j}$ for $0 < j \leq i \leq k$, is a complete set of invariants (of the class of $(V_1, \ldots, V_k)$).
(3) $\dim V_i \cap F_j$ for $0 < j \leq i \leq k$, is a complete set of invariants.

Moreover, there exists a bijection between $\mathrm{Flag}(r,s)/\mathcal{G}$ *and the set of integers $s_{i,j}$*

satisfying for $0 \le i \le k$ *the conditions*

$$
\begin{aligned}
s_i - s_{i-1} &= s_{i,1} + \cdots + s_{i,i} \\
r_i - r_{i-1} &= s_{i,i} + \cdots + s_{k,i}
\end{aligned}
$$

Example 2.6 Let $r = (2, 3, 4)$ and $s = (1, 2, 3)$. We write in a table the possible dimensions of the subspaces of the double filtration (1) and, besides, the canonical matrix representation of the corresponding canonical flag.

$$
(A) \quad \begin{array}{cccc} 1 & 1 & 1 & 2 \\ & 2 & 2 & 3 \\ & & 3 & 4 \end{array} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ & 1 & 0 \\ & & 1 \end{bmatrix} \qquad\qquad (B) \quad \begin{array}{cccc} 1 & 2 & 2 & 2 \\ & 2 & 2 & 3 \\ & & 3 & 4 \end{array} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ & 0 & 0 \\ & & 1 \end{bmatrix}
$$

$$
(C) \quad \begin{array}{cccc} 1 & 1 & 2 & 2 \\ & 2 & 3 & 3 \\ & & 3 & 4 \end{array} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ & 1 & 0 \\ & & 0 \end{bmatrix} \qquad\qquad (D) \quad \begin{array}{cccc} 1 & 2 & 2 & 2 \\ & 2 & 3 & 3 \\ & & 3 & 4 \end{array} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ & 0 & 1 \\ & & 0 \end{bmatrix}
$$

Now, let us consider a parametric family of generalized flags $(V_1(t), \ldots, V_k(t))$ of $(F_1, \ldots, F_k)$. More precisely, a smooth map

$$
M \to \mathrm{Flag}(r, s)
$$

with M a smooth contractible manifold (for example, an open neighbourhood of the origin in $\mathbb{R}^n$).

It is clear that if the invariants of $(V_1(t), \ldots, V_k(t))$ are constant for $t \in M$, we can find a smooth family $E_{i,j}(t)$ such that

$$
V_i(t) \cap F_j = E_{i,j}(t) \oplus (V_{i-1}(t) \cap F_j + V_i(t) \cap F_{j-1})
$$

(take, for example, $E_{i,j} = (V_{i-1}(t) \cap F_j + V_i(t) \cap F_{j-1})^{\perp} \cap V_i(t) \cap F_j$).

Thus, taking a smooth basis of $E_{i,j}(t)$, one can prove the following theorem

Theorem 2.7 *Let* $(V_1(t), \ldots, V_k(t))$ *be a smooth family of flags of the same class, as above, represented by* $X(t) \in \mathcal{M}$. *Then, there exists a smooth family of matrices* $P(t) \in \mathcal{G}$ *and* $Q(t) \in \Gamma$ *such that* $P(t)X(t)Q(t)$ *the canonical matrix (1).*

The next section consider the problem of how the class of $(V_1(t), \ldots, V_k(t))$ can change varying slightly the parameter t. In particular, we characterize the stable flags.

3 Versal deformation of a generalized flag

First we recall the definition of versal and miniversal deformation. Let M be a smooth manifold and x an element of M. A (local)deformation of $x \in M$ is a smooth map $\phi : \mathcal{U} \longrightarrow (M, x)$, where $\mathcal{U} \subset \mathbb{F}^n$ is a neighbourhood of the origin, (M, x) is a neighbourhood of the element $x \in M$, and $\phi(0) = x$. Let G be a Lie group acting smoothly on M on the left. We denote the action of $g \in G$ on $x \in M$ by $(g, x) \mapsto gx$.

Definition 3.1 *A local deformation of $x \in M$ is called* versal *(with regard to the action of G) if for any other deformation $\psi : \mathcal{V} \longrightarrow (M, x)$, $\mathcal{V} \subset \mathbb{F}^n$, $0 \in \mathcal{V}$, there exist a neighbourhood $\mathcal{V}' \subset \mathcal{V}$, $0 \in \mathcal{V}'$, a smooth map $h : \mathcal{V}' \longrightarrow \mathcal{U}$, with $h(0) = 0$ and a deformation $\theta : \mathcal{V}' \longrightarrow (G, I)$ of the identity element of G ($\theta(0) = I$) such that $\psi(v) = \theta(v)\varphi(h(v))$ for every $v \in \mathcal{V}'$. Versal deformations having a minimal number of parameters are called* miniversal.

Let $\mathcal{M}$ be an open and dense subset of a linear subvariety of $M_{p,q}$, $\mathcal{G}$ a subgroup of $\mathrm{Gl}(p)$ which is an open and dense subset of a linear subvariety of $M_{p,p}$ and Γ a subgroup of $\mathrm{Gl}(q)$ which is an open and dense subset of a linear subvariety of $M_{q,q}$. We suppose that $\mathcal{G}$ (respectively, Γ), acts on $\mathcal{M}$ on the left (respectively on the right) by matrix multiplication. We assume that the orbit space

$$\mathcal{M}/\Gamma := \{X\Gamma \,|\, X \in \mathcal{M}\}$$

has a differentiable structure such that the natural projection

$$\pi : \mathcal{M} \longrightarrow \mathcal{M}/\Gamma$$

is a submersion. In [3] it is proved the following result

Theorem 3.2 *With the above notation, a miniversal deformation of an orbit $X\Gamma$ in $\mathcal{M}/\Gamma$ is given by $(X + W)\Gamma$, $W \in \mathcal{W}$ where $\mathcal{W}$ is (a neighbourhood of the origin of) the set of matrices $W \in \overline{\mathcal{M}}$ such that*

$$\mathrm{trace}\,(PXW^*) = \mathrm{trace}\,(XQW^*) = 0$$

for all $P \in \overline{\mathcal{G}}$ and $Q \in \overline{\Gamma}$.

We apply now this theorem when $\mathcal{M}, \mathcal{G}$ and Γ are as in the previous sections. As usual, since the map $X \mapsto PXQ$ is a diffeomorphism, we can assume that X is a particular element of the orbit $\mathcal{G}X\Gamma$. So, let X be the canonical matrix (2) described in the last section. We are going to solve equations $\mathrm{trace}\,(PXW^*) = \mathrm{trace}\,(XQW^*) = 0$ in this case.

We have that

$$XQ = \begin{bmatrix} Q_{1,1} & Q_{1,2} & Q_{1,3} & \cdots \\ 0 & Q^1_{2,2} & Q^1_{2,3} & \cdots \\ 0 & 0 & Q^1_{3,3} & \\ 0 & 0 & 0 & \cdots \\ & Q^2_{2,2} & Q^2_{2,3} & \cdots \\ & 0 & Q^2_{3,3} & \\ & 0 & 0 & \cdots \\ & & Q^3_{3,3} & \\ & & 0 & \cdots \\ & & & \cdots \end{bmatrix}$$

where $Q_{i,j} = \begin{bmatrix} Q^1_{i,j} \\ Q^2_{i,j} \\ \cdots \end{bmatrix}$, with $Q^1_{i,j}$ the first $s_{i,1}$ rows of $Q_{i,j}$, etc.

and

$$PX = \begin{bmatrix} P^1_{1,1} & P^2_{1,1} & P^1_{1,2} & P^3_{1,1} & P^2_{1,2} & P^1_{1,3} & \cdots \\ & 0 & P^1_{2,2} & 0 & P^2_{2,2} & P^1_{2,3} & \cdots \\ & & & 0 & 0 & P^1_{3,3} & \cdots \\ & & & & & & \cdots \end{bmatrix}$$

where $P_{i,j} = \begin{bmatrix} P^1_{i,j} & P^2_{i,j} & \ldots \end{bmatrix}$, with $P^1_{i,j}$ the first $s_{1,1}$-columns of $P_{i,j}$, etc.

For our end it is crucial the following remark

Remark 3.3 Let Y be a matrix with a fixed zero structure and having in the rest of the entries different parameters. The set of all of theses matrices is a vector space of dimension the number of nonzero entries. We say that Z is a *complementary* matrix of Y if it is of the same size of Y and if Z has different parameters in the zero entries of Y and 0 in the nonzero entries of Y. It is clear that if Z is a complementary matrix of Y, then trace$YZ^* = 0$. Moreover, since the dimension of the set of complementary matrices of Y has complementary dimension to the set of matrices of the type of Y, we have that $\{Y\}^\perp$ is just the set of complementary matrices of Y.

Taking into account the last remark and the form of the matrices PX and XQ, we have that $\operatorname{trace}(PXW^*) = \operatorname{trace}(XQW^*) = 0$ implies that

(3) $\quad W = [W_{j,i}]$ with $W_{j,i} = 0$ if $i < j$ and otherwise, decomposing into blocks $W_{j,i}$ as $X_{j,i}$ in (2), we have

$$W_{j,i} = \begin{bmatrix} 0_{\beta_{i,j},\alpha_{i,j}} & 0 & 0 \\ 0 & 0_{s_{i,j}} & 0 \\ Y_{\overline{\beta}_{i,j},\alpha_{i,j}} & 0 & 0_{\overline{\beta}_{i,j},\overline{\alpha}_{i,j}} \end{bmatrix}$$

with $Y_{\overline{\beta}_{i,j},\alpha_{i,j}}$ a full parameter $\overline{\beta}_{i,j} \times \alpha_{i,j}$-matrix.

Thus, we have proved the following theorem

Theorem 3.4 *The set of flags $(X + W)\Gamma$ where X and W are as in (2) and (3), respectively, is a miniversal deformation of the flag $X\Gamma$.*

Let us write the first blocks of $X + W$

$$\left[\begin{array}{c|cc|ccc|c}
I & 0 & 0 & 0 & 0 & 0 & \dots \\
0 & I & 0 & 0 & 0 & 0 & \dots \\
0 & 0 & 0 & I & 0 & 0 & \dots \\
0 & 0 & 0 & 0 & 0 & 0 & \dots \\
\hline
 & 0 & I & 0 & 0 & 0 & \dots \\
 & * & 0 & 0 & I & 0 & \dots \\
 & * & 0 & * & 0 & 0 & \dots \\
\hline
 & & & 0 & 0 & I & \dots \\
 & & & * & * & 0 & \dots \\
\hline
 & & & & & & \dots
\end{array}\right]$$

Definition 3.5 *A flag $(V_1, \dots, V_k)$ is* stable *if there exists a neighbourhood U of $(V_1, \dots, V_k)$ in $\operatorname{Flag}(r, s)$ such that all the flags in U are equivalent.*

Of course, a necessary and sufficient condition for a flag to be stable is that the codimension of its orbit is 0. If $(V_1, \dots, V_k)$ is represented by $X \in \mathcal{M}$, the codimension of the orbit of $(V_1, \dots, V_k)$ is the same as the codimension of $\mathcal{G}X\Gamma$ in $\mathcal{M}$ (see [3]). Thus, theorem 3.4 has the following corollary.

Corollary 3.6 *The flag $(V_1, \dots, V_k)$ is stable if and only if $\overline{\beta}_{i,j} = 0$ for all $0 < j \leq i \leq k$. In particular, all the stable flags are equivalent.*

Example 3.7 In the example 2.6 the stable flags are the flags of the class (A). On the other side a miniversal deformation of the element

$$X = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ & 0 & 1 \\ & & 0 \end{bmatrix}$$

is given by

$$X + T_X(\mathcal{G}X\Gamma)^{\perp} = \{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ & x & 1 \\ & & y \end{bmatrix} \}$$

We see that the codimension of the orbit of $X\Gamma$ is 2 and it has tree adjacent classes of flags:

$$\begin{aligned} &(B) \text{ for } x = 0 \text{ and } y \neq 0 \\ &(C) \text{ for } x \neq 0 \text{ and } y = 0 \\ &(A) \text{ for } x \neq 0 \text{ and } y \neq 0 \end{aligned}$$

References

[1] U.Helmke, M.Shayman, *The biflag manifold and the fixed point set of a nilpotent transformation on the flag manifold* Linear Algebra Appl. 92 (1987), 125-159

[2] X.Puerta, U.Helmke, *The topology of the set of conditioned invariant subspaces* System and Control Letters, 40 (2000), 97–105

[3] F. Puerta, X. Puerta and S. Tarragona, *Versal deformations in orbit spaces* Linear Algebra Appl., In Press, Corrected Proof, Available online 18 July 2003,

[4] F. Puerta, X. Puerta, I. Zaballa, *A cellular decomposition of the maniofold of observable conditioned invariant subspaces* Linear Algebra Appl. 351-352 (2002), 601-621